where:

$$\sum_{j=1}^{N_1}\omega_{ij}y_{1j} = \sum_{j=1}^{N_1}\omega_{ij}\left(\mu_1 + \mathbf{x}_j\boldsymbol{\beta}_1 + e_{1j}\right) =$$
$$\mu_1\sum_{j=1}^{N_1}\omega_{ij} + \sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\boldsymbol{\beta}_1 + \sum_{j=1}^{N_1}\omega_{ij}e_{1j} = \quad (12)$$
$$\mu_1 + \left(\sum_{j=1}^{N_1}\omega_{ij}\mathbf{x}_j\right)\boldsymbol{\beta}_1 + \sum_{j=1}^{N_1}\omega_{ij}e_{1j}$$

and by developing ATE further using Eq. (11), we finally get the result in (10).

Proposition 2. *Formula of* ATE($\mathbf{x}_i$) *with neighbourhood interactions*. Given assumptions 2 and 3 and the result in proposition 1, we have that:

$$\text{ATE}(\mathbf{x}_i) = \text{ATE} + (\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + \sum_{j=1}^{N_1}\omega_{ij}(\bar{\mathbf{x}} - \mathbf{x}_j)\gamma\boldsymbol{\beta}_1 \quad (13)$$

where it is now easy to see that ATE $=\text{E}_{\mathbf{x}}\{\text{ATE}(\mathbf{x})\}$. The proof is in Appendix. See A2.

Proposition 3. *Baseline random-coefficient regression*. By substitution of equations (7) into the POM of Eq. (6), we obtain the following random-coefficient regression model (Wooldridge, 1997):

$$y_i = \eta + w_i \cdot \text{ATE} + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + w_i\sum_{j=1}^{N}\omega_{ij}w_j(\bar{\mathbf{x}} - \mathbf{x}_j)\gamma\boldsymbol{\beta}_1 + e_i \quad (14)$$

where, $\eta = \mu_0 + \gamma\mu_1 \quad \boldsymbol{\delta} = \boldsymbol{\beta}_1 - \boldsymbol{\beta}_0$

and

$$e_i = \gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} + e_{0i} + w_i(e_{1i} - e_{0i}) - w_i\gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j}$$

The proof is in Appendix. See A3.

Proposition 4. *Ordinary Least Squares (OLS) consistency*. Under assumption 1 (CMI), 2 and 3, the error tem of regression (14) has zero mean conditional on (w_i, $\mathbf{x}_i$), i.e.:

$$\text{E}\left(e_i | w_i, \mathbf{x}_i\right) = \text{E}\left(\gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} + e_{0i} + w_i(e_{1i} - e_{0i}) - w_i\gamma\sum_{j=1}^{N_1}\omega_{ij}e_{1j} \,\middle|\, w_i, \mathbf{x}_i\right) = 0 \quad (15)$$

thus implying that Eq. (14) is a regression model whose parameters can be *consistently* estimated by Ordinary Least Squares (OLS). The proof is in Appendix. See A4.

Once a consistent estimation of the parameters of (14) is obtained, we can estimate ATE directly from the regression, and ATE($\mathbf{x}_i$) by plugging the estimated parameters into formula (11). This is because ATE($\mathbf{x}_i$) becomes a function of consistent estimates, and thus consistent itself:

$$\text{plim ATE}(\mathbf{x}_i) = \text{ATE}(\mathbf{x}_i)$$

where $\text{ATE}(\mathbf{x}_i)$ is the plug-in estimator of ATE($\mathbf{x}_i$). Observe, however, that the (exogenous) weighting matrix $\boldsymbol{\Omega}=[\omega_{ij}]$ needs to be provided in advance.

Once the formulas for ATE and ATE($\mathbf{x}_i$) are available, it is also possible to recover the Average Treatment Effect on Treated (ATET) and on non-Treated (ATENT) as: